On a sharp volume estimate for gradient Ricci solitons with scalar curvature bounded below

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Abstract

In this note, we obtain a sharp volume estimate for complete gradient Ricci solitons with scalar curvature bounded below by a positive constant. Using Chen-Yokota's argument we obtain a local lower bound estimate of the scalar curvature for the Ricci flow on complete manifolds. Consequently, one has a sharp estimate of the scalar curvature for expanding Ricci solitons; we also provide a direct (elliptic) proof of this sharp estimate. Moreover, if the scalar curvature attains its minimum value at some point, then the manifold is Einstein.

Introduction

The Ricci flow $\frac{\partial}{\partial t}g(x,t) = -2Ric(x,t)$, was introduced by Hamilton in [6]. We say that a quadruple (M^n, g, f, ε), where (M^n, g) is a Riemannian manifold, f is a smooth function on M^n and $\varepsilon \in \mathbb{R}$, is a gradient Ricci soliton if

$$R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2} g_{ij} = 0. \tag{0.1}$$

We call f the potential function. We say that g is shrinking, steady, or expanding if $\varepsilon < 0$, $\varepsilon = 0$, or $\varepsilon > 0$, respectively.

The following volume growth estimate for complete shrinking gradient Ricci solitons was proved by O. Munteanu [8], with an important special case was proved by H.-D. Cao and D.-T. Zhou [5]. Let $(M^n, g, f, -1)$ be a complete shrinking gradient Ricci soliton. Given $o \in M^n$, there exists a constant $C < \infty$ such that

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$$V(B(o,r)) \leq C(r+1)^n$$

for all $r \geq 0$, where $B(o,r)$ is the ball of radius r at center o and $V(B(o,r))$ denotes the volume of $B(o,r)$ with respect to the metric g. From the proof of Proposition 2.1 in [4], we can obtain the following property; see Lemma 1.1 below.

Lemma 0.1 *Let $(M^n, g, f, -1)$ be a complete gradient shrinking Ricci soliton with $R \geq \delta > 0$. Then for any $\eta > 0$, there exists a constant $C_1 < \infty$ depending on η and the soliton such that*

$$V(B(o,r)) \leq C_1(r+1)^{n-(2-\eta)\delta}.$$

for all $r > 0$.

We sharpen the above result as follows, which is our main theorem.

Theorem 0.2 *Let $(M^n, g, f, -1)$ be a complete shrinking gradient Ricci soliton with $R \geq \delta > 0$. Then given $o \in M^n$, there exists a constant $C < \infty$ depending only on δ, o and the soliton such that*

$$V(B(o,r)) \leq C(r+1)^{n-2\delta}$$

for all $r \geq 0$.

Remark 0.3 *Above result is sharp. For example, a product $M^n = N^k \times \mathbb{R}^{n-k} (k = 2, 3, \cdots, n)$, where N^k is an Einstein manifold with constant scalar curvature $\frac{k}{2}$ and take $f = \frac{|x|^2}{4}$ on $\mathbb{R}^{n-k}$, then the equality in Theorem 0.2 holds.*

The following property for gradient Ricci solitons is the second part of Theorem 1.3 in Z.-H. Zhang [11]. In fact, part 1 is a consequence of Corollary 2.5 in B.-L. Chen [2].

Theorem 0.4 *Let (M^n, g, f, ε) be a noncompact complete gradient Ricci soliton.*

1. *If the gradient soliton is shrinking or steady, then $R \geq 0$.*
2. *If the gradient soliton is expanding, then there exists a positive constant $C(n)$ such that $R \geq -C(n)\varepsilon$.*

The following property is an improvement of part 2 of Theorem 0.4, which is the sharp estimate for noncompact expanding gradient Ricci solitons. The compact case follows from a direct application of the maximum principle; we know the manifold is Einstein (see Proposition 9.43 in [3]).

Theorem 0.5 *Let $(M^n, g, f, 1)$ be a complete expanding gradient Ricci soliton. Then $R \geq -\frac{n}{2}$. Furthermore, if there exists a point $x_0 \in M^n$ such that $R(x_0) = -\frac{n}{2}$, then (M^n, g) is Einstein, i.e., $R_{ij} = -\frac{1}{2} g_{ij}$.*

The first part of Theorem 0.5 is a consequence of Corollary 2.3 (i) in B.-L. Chen [2](see Corollary 2.2 below); we also provide a direct (elliptic) proof of this sharp estimate.

1 Volume growth of complete noncompact gradient Ricci solitons

We consider the complete shrinking gradient Ricci solitons in this section, i.e. $\varepsilon = -1$. Normalizing f, from (0.1) we have

$$R + |\nabla f|^2 - f \equiv 0. \tag{1.1}$$

Define

$$\mathbf{V}(c) \doteqdot \textstyle\int_{\{f \le c\}} d\mu = Vol\{f \le c\},$$

$$\mathbf{R}(c) \doteqdot \textstyle\int_{\{f \le c\}} R d\mu.$$

By the co-area formula

$$\mathbf{V}'(c) = \textstyle\int_{\{f=c\}} \frac{1}{|\nabla f|} d\sigma,$$

$$\mathbf{R}'(c) = \textstyle\int_{\{f=c\}} \frac{R}{|\nabla f|} d\sigma.$$

Since $R \ge 0$ (see Corollary 2.3 below), we have

$$\mathbf{R}(c) \ge 0 \text{ and } \mathbf{R}'(c) \ge 0.$$

Integrating $R + \Delta f = \frac{n}{2}$ over $\{f \le c\}$ yields

$$\begin{aligned} \frac{n}{2}\mathbf{V}(c) - \mathbf{R}(c) &= \int_{\{f \le c\}} \Delta f d\mu \\ &= \int_{\{f=c\}} \frac{\partial f}{\partial \nu} d\sigma \\ &= \int_{\{f=c\}} |\nabla f| d\sigma. \end{aligned} \tag{1.2}$$

In particular,

$$\frac{n}{2}\mathbf{V}(c) \ge \mathbf{R}(c). \tag{1.3}$$

Since by (1.1),

$$|\nabla f| = \tfrac{|\nabla f|^2}{|\nabla f|} = \tfrac{f-R}{|\nabla f|}.$$

So we have

$$\frac{n}{2}\mathbf{V}(c) - \mathbf{R}(c) = c\mathbf{V}'(c) - \mathbf{R}'(c). \tag{1.4}$$

That is

$$\frac{n}{2}\mathbf{V}(c) - c\mathbf{V}'(c) = \mathbf{R}(c) - \mathbf{R}'(c). \tag{1.5}$$

Lemma 1.1 *Let $(M^n, g, f, -1)$ be a complete gradient shrinking Ricci soliton with $R \geq \delta > 0$, then for any $\eta > 0$, there exist c_0, C_1 depending on η and δ, when $c \geq c_0$, we have*

$$\boldsymbol{V}(c) \leq C_1 c^{\frac{n-(2-\eta)\delta}{2}}.$$

Proof. For the sake of completeness, we provide the detailed proof. The proof is similar to the proof of Theorem 1 in [8](or the proof of Proposition 2.1 in [4]). If $\eta \geq 2$, this has done by Theorem 1 in [8].

So we only consider $\eta < 2$. Now using the positive lower bound for R and $\eta < 2$, we have

$$\begin{aligned} \frac{n-(2-\eta)\delta}{2}\mathbf{V}(c) - \frac{\eta}{2}\mathbf{R}(c) \geq & \frac{n}{2}\mathbf{V}(c) - \mathbf{R}(c) \\ = & c\mathbf{V}'(c) - \mathbf{R}'(c). \end{aligned} \tag{1.6}$$

This implies

$$\begin{aligned} \frac{d}{dc}(c^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(c)) = \quad & c^{-\frac{n+2-(2-\eta)\delta}{2}}(c\mathbf{V}'(c) - \tfrac{n-(2-\eta)\delta}{2}\mathbf{V}(c)) \\ \leq \quad & c^{-\frac{n+2-(2-\eta)\delta}{2}}(\mathbf{R}'(c) - \tfrac{\eta}{2}\mathbf{R}(c)). \end{aligned}$$

Integrating this by parts on $[c_0, \overline{c}]$ yields

$$\begin{aligned} \overline{c}^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(\overline{c}) - c_0^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(c_0) \leq \quad & \overline{c}^{-\frac{n+2-(2-\eta)\delta}{2}}\mathbf{R}(\overline{c}) - c_0^{-\frac{n+2-(2-\eta)\delta}{2}}\mathbf{R}(c_0) \\ & + \int_{c_0}^{\overline{c}}(\tfrac{n+2-(2-\eta)\delta}{2} - \tfrac{\eta c}{2})c^{-\frac{n+4-(2-\eta)\delta}{2}}\mathbf{R}(c)dc. \end{aligned}$$

Since $\mathbf{R}(c) \geq 0$, for $c_0 \geq \frac{n+2-(2-\eta)\delta}{\eta}$ we have

$$\begin{aligned} \overline{c}^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(\overline{c}) - c_0^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(c_0) \leq \quad & \overline{c}^{-\frac{n+2-(2-\eta)\delta}{2}}\mathbf{R}(\overline{c}) \\ \leq \quad & \tfrac{n}{2}\overline{c}^{-\frac{n+2-(2-\eta)\delta}{2}}\mathbf{V}(\overline{c}) \end{aligned}$$

the last inequality has used (1.3). Thus if $\overline{c} \geq \max\{n, c_0\}, c_0 \geq \frac{n+2-(2-\eta)\delta}{\eta}$, then

$$\mathbf{V}(\overline{c}) \leq 2c_0^{-\frac{n-(2-\eta)\delta}{2}}\mathbf{V}(c_0)\overline{c}^{\frac{n-(2-\eta)\delta}{2}}.$$

So Lemma holds. ∎

Theorem 1.2 *Let $(M^n, g, f, -1)$ be a complete gradient shrinking Ricci soliton with $R \geq \delta > 0$, then there exists a positive constant C depending only on δ, o and the soliton such that*

$$V(B(o, r)) \leq C(1 + r)^{n-2\delta}.$$

Proof. Since $R \geq \delta, \mathbf{R}(c) \geq 0$, we have

$$\begin{aligned} \frac{n-2\delta}{2}\mathbf{V}(c) \geq & \quad \tfrac{n}{2}\mathbf{V}(c) - \mathbf{R}(c) \\ = & \quad c\mathbf{V}'(c) - \mathbf{R}'(c). \end{aligned}$$

Since $\mathbf{R}'(c) \geq 0$, we have

$$\begin{aligned} \frac{d}{dc}(c^{-\frac{n-2\delta}{2}}\mathbf{V}(c)) = & \quad c^{-\frac{n+2-2\delta}{2}}(c\mathbf{V}'(c) - \tfrac{n-2\delta}{2}\mathbf{V}(c)) \\ \leq & \qquad\qquad c^{-\frac{n+2-2\delta}{2}}\mathbf{R}'(c). \end{aligned}$$

Integrating this by parts on $[c_0, \overline{c}]$ yields

$$\begin{aligned} \overline{c}^{-\frac{n-2\delta}{2}}\mathbf{V}(\overline{c}) - c_0^{-\frac{n-2\delta}{2}}\mathbf{V}(c_0) \leq & \int_{c_0}^{\overline{c}} c^{-\frac{n+2-2\delta}{2}}\mathbf{R}'(c)dc \\ = & \overline{c}^{-\frac{n+2-2\delta}{2}}\mathbf{R}(\overline{c}) - c_0^{-\frac{n+2-2\delta}{2}}\mathbf{R}(c_0) \\ & + \frac{n+2-2\delta}{2}\int_{c_0}^{\overline{c}} c^{-\frac{n+4-2\delta}{2}}\mathbf{R}(c)dc. \end{aligned} \tag{1.7}$$

By (1.3), we have

$$\int_{c_0}^{\overline{c}} c^{-\frac{n+4-2\delta}{2}}\mathbf{R}(c)dc \leq \tfrac{n}{2}\int_{c_0}^{\overline{c}} c^{-\frac{n+4-2\delta}{2}}\mathbf{V}(c)dc.$$

Let $\eta = \frac{1}{\delta}$ in Lemma 1.1, so when c is large enough, we have

$$\mathbf{V}(c) \leq C_1 c^{\frac{n+1-2\delta}{2}}.$$

So

$$\begin{aligned} \int_{c_0}^{\overline{c}} c^{-\frac{n+4-2\delta}{2}}\mathbf{R}(c)dc \leq & \quad \tfrac{nC_1}{2}\int_{c_0}^{\overline{c}} c^{-\frac{n+4-2\delta}{2}}c^{\frac{n+1-2\delta}{2}}dc \\ = & \qquad \tfrac{nC_1}{2}\int_{c_0}^{\overline{c}} c^{-\frac{3}{2}}dc \\ = & \qquad nC_1(c_0^{-\frac{1}{2}} - \overline{c}^{-\frac{1}{2}}) \\ \leq & \qquad nC_1 c_0^{-\frac{1}{2}}. \end{aligned}$$

Since (1.3) and $\mathbf{V}(c) \geq 0$, $\delta \leq \frac{n}{2}$. So

$$\overline{c}^{-\frac{n-2\delta}{2}}\mathbf{V}(\overline{c}) - c_0^{-\frac{n-2\delta}{2}}\mathbf{V}(c_0) \leq \frac{n}{2}\overline{c}^{-\frac{n+2-2\delta}{2}}\mathbf{V}(\overline{c}) + \frac{n+2-2\delta}{2}nC_1 c_0^{-\frac{1}{2}}. \tag{1.8}$$

Then same argument in the proof of Lemma 1.1, when $\overline{c}$ is large enough, there exists a constant C_2 depending only on δ such that

$$\mathbf{V}(\overline{c}) \leq C_2\overline{c}^{\frac{n-2\delta}{2}}. \tag{1.9}$$

By Theorem 1.1 of [5] (or see [1]), there exists a constant C depending only on g and o such that

$$\frac{1}{4}(r(x) - C)^2 \leq f(x) \leq \frac{1}{4}(r(x) + C)^2 \tag{1.10}$$

where $r(x)$ denotes the distance from x to o. Hence we obtain the result. ■

2 Lower bound of scalar curvature for Ricci flow

In this section, we observe that by a modification of B.-L. Chen's theorem (Corollary 2.3(i) in [2]), we obtain a local lower bound estimate of the scalar curvature for the Ricci flow on complete manifolds, we follow Yokota's argument in Proposition A.3 in [10].

Theorem 2.1 *For any $0 < \varepsilon < \frac{2}{n}$. Suppose $(M^n, g(t))$, $t \in [\alpha, \beta]$ is a complete solution to Ricci flow, $p \in M$, then there exist constants $C(p)$ depending on p and the metrics $g(t)(t \in [\alpha, \beta])$ and C such that when $c \geq C(p)$, we have*

$$R(x,t) \geq -B\frac{e^{2AB(t-\alpha)}+1}{e^{2AB(t-\alpha)}-1} \tag{2.1}$$

whenever $x \in B_{g(t)}(p,c), t \in (\alpha, \beta]$, where $A(\varepsilon) = \frac{2}{n} - \varepsilon, B(\varepsilon) = \frac{3C}{2\sqrt{A\varepsilon c^2}}$.

Proof. First we use the cutoff function in the proof of Proposition A.3 in [10]. Let $\eta : \mathbb{R} \to [0,1]$ be a nonincreasing $\mathcal{C}^2$ function such that $|\eta'|$ and $|\eta''|$ are bounded and $\eta(u) = 1$ for any $u \in (-\infty, 1]$, $\eta(u) = 0$ for any $u \in [2, \infty)$ and $\eta(u) = (2-u)^4$ for any $u \in [\frac{3}{2}, 2]$. Then there exists a positive constant C such that

$$\frac{(\eta'(u))^2}{\eta(u)} \leq C\eta(u)^{\frac{1}{2}}, \tag{2.2}$$

$$|\eta''(u)| \leq C\eta(u)^{\frac{1}{2}}. \tag{2.3}$$

Clearly we can choose a number $r_0 \in (0,1)$, such that

$$Rc(g(t)) \leq (n-1)r_0^{-2}$$

in $B_{g(t)}(p, r_0)$ for $t \in [\alpha, \beta]$. Let $C(p) = r_0 + \frac{5}{3}(n-1)r_0^{-1}(\beta - \alpha)$ and given any $c \geq C(p)$.

Given any time $t_0 \in (\alpha, \beta]$ and suppose that

$$R(p, t_0) < 0. \tag{2.4}$$

Define $Q : M \times [\alpha, t_0] \to \mathbb{R}$ by

$$Q(x,t) = \eta(\frac{\tilde{r}(x,t)}{c})R(x,t), \tag{2.5}$$

where

$$\tilde{r}(x,t) \doteqdot d_{g(t)}(x,p) + \frac{5}{3}(n-1)r_0^{-1}(t_0 - t). \tag{2.6}$$

Then $Q(x,t)$ is a compactly support function.

By Lemma 8.3 (a) in [9], we have

$$(\frac{\partial}{\partial t} - \Delta)\tilde{r}(x,t) \geq 0 \tag{2.7}$$

whenever $d_{g(t)}(x,p) > r_0, \quad t \in [\alpha, t_0]$, in the barrier sense. We have

$$\begin{aligned}(\frac{\partial}{\partial t}-\Delta)Q = \quad & \eta(\frac{\partial}{\partial t}-\Delta)R + \frac{\eta'}{c}R(\frac{\partial}{\partial t}-\Delta)\tilde{r} \\ & -\frac{2}{c}\eta' < \nabla\tilde{r}, \nabla R > -\frac{\eta''}{c^2}R\end{aligned}$$

where η denotes $\eta(\frac{\tilde{r}}{c})$. In the case of $d_{g(t)}(x,p) \le r_0$, then $\tilde{r}(x,t) \le c$, so at point (x,t), $\eta = 1, \quad \eta' = \eta'' = 0$, so

$$(\frac{\partial}{\partial t}-\Delta)Q = (\frac{\partial}{\partial t}-\Delta)R. \tag{2.8}$$

In the case of $d_{g(t)}(x,p) > r_0$, then we applying (2.7) and $\eta' \le 0$, we have at a point where $R \le 0$

$$(\frac{\partial}{\partial t}-\Delta)Q \ge \eta\frac{2}{n}R^2 - \frac{2\eta'}{c\eta} < \nabla\tilde{r}, \nabla Q > +\frac{1}{c^2}(2\frac{(\eta')^2}{\eta} - \eta'')R \tag{2.9}$$

whenever $\eta \ne 0$. Hence by both of cases, we have at point (x,t) where $R(x,t) \le 0$, (2.9) holds whenever $\eta \ne 0$. Applying (2.2) and (2.3) to (2.9), we have at any point where $Q < 0$

$$(\frac{\partial}{\partial t}-\Delta)Q \ge \eta\frac{2}{n}R^2 - \frac{2\eta'}{c\eta} < \nabla\tilde{r}, \nabla Q > +\frac{3C}{c^2}\eta^{\frac{1}{2}}R. \tag{2.10}$$

Let $Q_m(t) = \min_{x\in M} Q(x,t)$. Then

$$Q_m(t_0) \le R(p, t_0) < 0.$$

By (2.10) we have for any $t \in [\alpha, t_0]$ where $Q_m(t) < 0$ and for any x_t such that $Q(x_t, t) = Q_m(t)$, then for any $\varepsilon \in (0, \frac{2}{n})$

$$\begin{aligned}\frac{d_-}{dt}Q_m(t) \ge \quad & \frac{2}{n}\eta R(x_t,t)^2 + \frac{3C}{c^2}\eta^{\frac{1}{2}}R(x_t,t) \\ \ge \quad & (\frac{2}{n}-\varepsilon)\eta R(x_t,t)^2 - \frac{9C^2}{4\varepsilon C^4} \\ \ge \quad & (\frac{2}{n}-\varepsilon)Q_m^2 - \frac{9C^2}{4\varepsilon c^4}\end{aligned}$$

using $ab \ge -\varepsilon a^2 - \frac{1}{4\varepsilon}b^2$ and $0 < \eta \le 1$.

Recall that the solution of ODE

$$\begin{cases}\frac{dq}{dt} & = A(q^2 - B^2), \\ q(t_0) & = q_0\end{cases}$$

on $[\alpha, t_0]$, then

$$q(t) = \begin{cases}-B\frac{De^{-2AB(t_0-t)}+1}{De^{-2AB(t_0-t)}-1} & \text{if } B \ne -q_0, \\ q_0 & \text{if } B = -q_0\end{cases}$$

where $D = \frac{q_0 - B}{q_0 + B}$ provided $B \ne -q_0$.

Taking $A = \frac{2}{n} - \varepsilon, \quad B = \frac{3C}{2\sqrt{A\varepsilon}c^2}$, and $q_0 = Q_m(t_0) < 0$, then we have

$$Q_m(t) \le q(t).$$

$q(t) > -\infty$ for $t \in [\alpha, t_0]$, since $Q_m(t) > -\infty$ for $t \in [\alpha, t_0]$.

Case (1). If $q_0 \ge -B$, then we have

$$R(x, t_0) \ge Q_m(t_0) \ge -B \tag{2.11}$$

whenever $x \in B_{g(t_0)}(p, c)$, since $\eta = 1$.

Case (2). If $q_0 < -B$, then $D > 1$ and since $q(t) > -\infty$ for $t \in [\alpha, t_0]$, we have

$$De^{-2AB(t_0-t)} - 1 > 0$$

for all $t \in [\alpha, t_0]$, so

$$De^{-2AB(t_0-\alpha)} - 1 > 0 \text{ i.e., } q_0 > -B\tfrac{e^{2AB(t_0-\alpha)}+1}{e^{2AB(t_0-\alpha)}-1},$$

so that

$$R(x, t_0) \ge q_0 > -B\frac{e^{2AB(t_0-\alpha)}+1}{e^{2AB(t_0-\alpha)}-1} \tag{2.12}$$

whenever $x \in B_{g(t_0)}(p, c)$.

Since $B > 0$, (2.12) is a better estimate, we conclude that (2.12) holds in either case.

Since c independent of t, we complete the proof of this theorem. ■

Hence we also obtain a consequence of Corollary 2.3(i) in B.-L. Chen [2].

Corollary 2.2 *Suppose $(M^n, g(t))$, $t \in [\alpha, \beta]$, is a complete solution to Ricci flow, then*

$$R \ge -\frac{n}{2(t-\alpha)} \tag{2.13}$$

on $M \times (\alpha, \beta]$.

Proof. For any $t_0 \in (\alpha, \beta]$. Now fix $\varepsilon \in (0, \frac{2}{n})$ and let $c \to \infty$. Then $B \to 0$. Since

$$\lim_{B\to 0} B\frac{e^{2AB(t_0-\alpha)}+1}{e^{2AB(t_0-\alpha)}-1} = \frac{1}{A(t_0-\alpha)}$$

and (2.12) independent of c, we obtain

$$R(x, t_0) \ge -\tfrac{1}{(\frac{2}{n}-\varepsilon)(t_0-\alpha)}.$$

on $M \times \{t_0\}$. Finally, taking $\varepsilon \to 0$, we obtain

$$R(x, t_0) \ge -\frac{n}{2(t_0-\alpha)}.$$

on $M \times \{t_0\}$. Since above argument holds for any $t \in (\alpha, \beta]$, we obtain the corollary. ■

The following property is Corollary 2.5 in [2](or Proposition A.3 in [10]).

Corollary 2.3 *If $(M^n, g(t))$, $\quad t \in (-\infty, 0]$, is a complete ancient solution to the Ricci flow, then*

$$R \geq 0$$

on $M \times (-\infty, 0]$.

Corollary 2.4 *Suppose (M^n, g, f, ε) be a noncompact complete gradient Ricci soliton. Then*

1. *If the gradient soliton is shrinking or steady, then $R \geq 0$.*
2. *If the gradient soliton is expanding, then $R \geq -\frac{n\varepsilon}{2}$. Moreover, if the scalar curvature attain the minimum value $-\frac{n\varepsilon}{2}$ at some point, then $(M^n, g(t))$ is Einstein.*

Proof. Part (1) is a consequence of Corollary 2.3.

As shown in Theorem 4.1 of [3], associated to the metric and the the potential function f, there exists a family of metrics $g(t)$, a solution to Ricci flow $\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$, with the property that $g(0) = g$, and a family of diffeomorphisms $\phi(t)$, which is generated by the vector field $X = \frac{1}{\tau}\nabla f$, such that $\phi(0) = id$, and $g(t) = \tau(t)\phi^*(t)g$ with $\tau(t) = 1 + \varepsilon t > 0$, as well as $f(x,t) = \phi^*(t)f(x)$.

For expanding gradient Ricci soliton, i.e. $\varepsilon > 0$. We know $t \in (-\frac{1}{\varepsilon}, \infty)$, so by Corollary 2.2 we obtain $R(x,t) \geq -\frac{n}{2(t+\frac{1}{\varepsilon})}$, i.e., $R(x,t) \geq -\frac{n\varepsilon}{2\tau}$.

Let $\tilde{R} = R + \frac{n\varepsilon}{2\tau}$, so $\tilde{R} \geq 0$ and

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{R} =& \Delta R + 2|Rc|^2 - \frac{n\varepsilon^2}{2\tau^2} \\
=& \Delta\tilde{R} + 2|Rc - \frac{R}{n}g|^2 + \frac{2R^2}{n} - \frac{n\varepsilon^2}{2\tau^2} \\
=& \Delta\tilde{R} + 2|Rc - \frac{R}{n}g|^2 + \frac{2}{n}\tilde{R}(\tilde{R} - \frac{n\varepsilon}{\tau}) \\
\geq& \Delta\tilde{R} - \frac{2\varepsilon}{\tau}\tilde{R}.
\end{aligned} \tag{2.14}$$

So

$$\frac{\partial}{\partial t}(\tau^2\tilde{R}) \geq \Delta(\tau^2\tilde{R}). \tag{2.15}$$

By strong maximum principle (Theorem 6.54 in [3]), we know that if there exists a point x_0 such that $\tilde{R}(x_0, t_0) = 0$ for some $t_0 > -\frac{1}{\varepsilon}$, then $\tilde{R}(x, t_0) \equiv 0$ for all $x \in M$. So

$$R(x, t_0) \equiv -\tfrac{n\varepsilon}{2\tau(t_0)}.$$

So when $t_0 = 0$, i.e. $\tau(0) = 1$, then $R(x, 0) \equiv -\frac{n\varepsilon}{2}$. From Lemma 3.1 (1) below (or see section 4.1 in [3]), we have

$$\Delta R + 2|Rc|^2 - < \nabla f, \nabla R > + \varepsilon R = 0. \tag{2.16}$$

So by (2.16) we know

$$|Rc + \frac{\varepsilon}{2} g|^2 = 0.$$

So

$$R_{ij} = -\frac{\varepsilon}{2} g_{ij}.$$

Hence (M, g, f, ε) is Einstein. ■

In next section, we will provide a direct (elliptic) proof of the first part of Theorem 0.5.

3 Direct proof for expanding Ricci solitons

In this section, we provide a direct (elliptic) proof of lower bound for scalar curvature for complete expanding gradient Ricci solitons. We use a cutoff function argument to equation (0.1).

Lemma 3.1 *Let (M^n, g, f, ε) be a complete gradient Ricci soliton. Fix $o \in M^n$, and define $r(x) \doteqdot d(x, o)$, then the following hold*

1. $\Delta R + 2|Rc|^2 - < \nabla f, \nabla R > + \varepsilon R = 0.$
2. *Suppose $Ric \leq (n-1)K$ on $B(o, r_0)$, for some positive numbers r_0 and K. Then for any point x, outside $B(o, r_0)$*

$$(\Delta r - < \nabla f, \nabla r >)(x) \leq - < \nabla f, \nabla r > (o) + \frac{\varepsilon}{2} r(x) + (n-1)\{\frac{2}{3} K r_0 + r_0^{-1}\}.$$

Part (1) is well known. Part (2) follows from an idea of Perelman; see Lemma 8.3 in [9] and its antecedent in §17 on 'Bounds on changing distances', in [7]. For the detailed proof of part 2, also see [1].

Now we prove the first part of Theorem 0.5.

Proof. For expanding gradient Ricci solitons, let $\varepsilon = 1$, so that (1) in Lemma 1.1 is

$$\Delta R + 2|Rc|^2 - < \nabla f, \nabla R > + R = 0. \tag{3.1}$$

If M^n were closed, by Proposition 9.43 in [3], we know $(M^n, g, f, 1)$ is Einstein. So we only consider the noncompact case. Fix $o \in M^n$ and fix a large number b. Let

$$\eta : [0, \infty) \to [0, 1]$$

be a $\mathcal{C}^\infty$ nonincreasing cutoff function with $\eta(u) = 1$ for $u \in [0, 1]$ and $\eta(u) = 0$ for $u \in [1 + b, \infty)$. Define $\Phi : M \to \mathbb{R}$ by

$$\Phi(x) = \eta(\frac{r(x)}{c}) R(x)$$

for $c \in (0, \infty)$. Later we shall take $c \to \infty$.

We have

$$\Delta \Phi = \eta \Delta R + \frac{2\eta'}{c} < \nabla r, \nabla R > + (\frac{\eta'}{c} \Delta r + \frac{\eta''}{c^2}) R.$$

We have dropped $' \circ \frac{r}{c}'$ in our notation. By (3.1), we have

$$\begin{aligned}\Delta\Phi =&\eta(-2|Rc|^2 + < \nabla f, \nabla R > - R) + \frac{2}{c}\eta' < \nabla r, \nabla R > + (\frac{\eta'}{c}\Delta r + \frac{\eta''}{c^2})R \\ =& < \nabla f, \nabla \Phi > + \frac{2}{c}\frac{\eta'}{\eta} < \nabla r, \nabla\Phi > - (\frac{2}{c^2}\frac{(\eta')^2}{\eta} + \frac{\eta'}{c} < \nabla f, \nabla r >)R \\ &+ \eta(-2|Rc|^2 - R) + (\frac{\eta'}{c}\Delta r + \frac{\eta''}{c^2})R\end{aligned} \tag{3.2}$$

at all points where $\eta \neq 0$.

Suppose $x_0 \in M$ is such that

$$\Phi(x_0) = \min_M \Phi < 0. \tag{3.3}$$

Since $R(x_0) < 0$, at x_0 we have

$$0 \geq \eta(-\frac{2}{n}R - 1) + \frac{\eta'}{c}(\Delta r - < \nabla f, \nabla r >) + \frac{1}{c^2}(\eta'' - 2\frac{(\eta')^2}{\eta}). \tag{3.4}$$

We consider two cases, depending on the location of x_0.

Case (i). Suppose $r(x_0) < c$, so that $\eta(\frac{r(x_0)}{c}) = 1$ in a neighborhood of x_0. Then (3.4) and (3.3) imply

$$\begin{aligned} 0 \geq & -\tfrac{2}{n}R(x_0) - 1 \\ = & -\tfrac{2}{n}\Phi(x_0) - 1 \\ \geq & -\tfrac{2}{n}\eta(\tfrac{r(x)}{c})R(x) - 1 \end{aligned}$$

for all $x \in M$. This implies the desired estimate

$$R(x) \geq -\frac{n}{2} \tag{3.5}$$

for all $x \in B(o,c)$ since $\eta(\frac{r}{c}) = 1$ in $B(o,c)$.

Case (ii). Now suppose $r(x_0) \geq c$ and again consider (3.4). Note that we may choose η so that

$$\eta'' - 2\frac{(\eta')^2}{\eta} \geq -C_2 \tag{3.6}$$

for some universal constant $C_2 < \infty$. Since $\eta' \leq 0$, applying Lemma 3.1 (2) and (3.6) to (3.4) yields for all $x \in M$

$$\begin{aligned}\frac{2}{n}\Phi(x) \geq& \frac{2}{n}\Phi(x_0) \\ \geq& \frac{\eta'(\frac{r(x_0)}{c})}{c}(\frac{n-1}{r_0} - < \nabla f, \nabla r > (o) + \frac{1}{2}r(x_0) + \frac{2}{3}r_0 \max_{B(o,r_0)} Rc) \\ &- \eta(\frac{r(x_0)}{c}) - \frac{C_2}{c^2}\end{aligned} \tag{3.7}$$

where C_2 independent of c. Taking $r_0 = 1$ and $c \geq 2$, we have for all $x \in B(o,c)$

$$\begin{aligned}\frac{2}{n}R(x) \geq & \frac{\eta'(\frac{r(x_0)}{c})}{c}(n-1+|\nabla f|(o)+\frac{1}{2}r(x_0)+\frac{2}{3}\max_{B(o,1)} Rc) \\ & -\eta(\frac{r(x_0)}{c})-\frac{C_2}{c^2}\end{aligned} \tag{3.8}$$

Since $-C_2 \leq \eta' \leq 0$ imply that for all $x \in B(o,c)$

$$\begin{aligned}\frac{2}{n}R(x) \geq & -\frac{C_2}{c}(n-1+|\nabla f|(o)+\frac{2}{3}\max_{B(o,1)} Rc+\frac{1}{c}) \\ & +\frac{1}{2}\eta'(\frac{r(x_0)}{c})\frac{r(x_0)}{c}-\eta(\frac{r(x_0)}{c}).\end{aligned} \tag{3.9}$$

When take $c \to \infty$, then the first term of right hand side of (3.9) tends to 0. So we only consider to estimate the term $\frac{1}{2}\eta'(\frac{r(x_0)}{c})\frac{r(x_0)}{c} - \eta(\frac{r(x_0)}{c})$. Since $x_0 \in B(o,(1+b)c) - B(o,c)$, we have $1 \leq \frac{r(x_0)}{c} < 1+b$. Define $h_\eta(u)$ by

$$h_\eta(u) = \tfrac{1}{2}\eta'(u)u - \eta(u).$$

So we only estimate $h_\eta(u)$ for $u \in [1, 1+b]$.

If we replace η with nonnegative piecewise linear function $\theta(u)$ such that

$$\theta(u) = \begin{cases} 1 & \text{if } u \in [0,1], \\ \frac{1+b-u}{b} & \text{if } u \in [1,1+b], \\ 0 & \text{if } u \in [1+b,\infty) \end{cases}$$

then $h_\theta(u) = -\frac{2b+2-u}{2b}$ for $u \in [1,1+b]$. So $h_\theta(u) \geq -1$ for $u \in [2,b]$ and $h_\theta(u) \geq -1-\frac{1}{b}$ for $u \in [1,2]$. For any small positive number δ, we can obtain a $\mathcal{C}^\infty$ cutoff function β after smooth the linear function θ such that $\beta(u) = \theta(u)$ for $u \in [0,1] \cup [2,b] \cup [1+b,\infty)$ and $-\frac{1+\delta}{b} \leq \beta'(u) \leq 0$ for $u \in [1,2] \cup [b,1+b]$. So when b is large and $\delta \leq \frac{b-1}{b+1}$, we have $h_\beta(u) \geq -1-\frac{1+\delta}{b}$ for $u \in [1,1+b]$. Let η equal β, take $c \to \infty, \delta \to 0, b \to \infty$, by (3.9) we obtain

$$\tfrac{2}{n}R(x) \geq -1$$

for all $x \in M$. So

$$R(x) \geq -\frac{n}{2} \tag{3.10}$$

for all $x \in M$. ■

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